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2001 J. Phys. A: Math. Gen. 34 8119

(http://iopscience.iop.org/0305-4470/34/39/311)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 8119–8133 PII: S0305-4470(01)25095-1

Realizations of the Lie superalgebra $q(2)$ and applications

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Received 3 May 2001
Published 21 September 2001
Online at stacks.iop.org/JPhysA/34/8119

Abstract
The Lie superalgebra $q(2)$ and its class of irreducible representations V_p of dimension $2p$ (p being a positive integer) are considered. The action of the $q(2)$ generators on a basis of V_p is given explicitly, and from here two realizations of $q(2)$ are determined. The $q(2)$ generators are realized as differential operators in one variable x, and the basis vectors of V_p as 2-arrays of polynomials in x. Following such realizations, it is observed that the Hamiltonian of certain physical models can be written in terms of the $q(2)$ generators. In particular, the models given here as an example are the sphaleron model, the Moszkowski model and the Jaynes–Cummings model. For each of these, it is shown how the $q(2)$ realization of the Hamiltonian is helpful in determining the spectrum.

PACS numbers: 02.20.−v, 11.30.Pb, 42.50.−p

1. Introduction

Since their introduction in supersymmetry [1–3], Lie superalgebras and their irreducible representations (simple modules) have been the subject of much attention in both mathematical [4–6] and physics literature, where both finite [7–9] and infinite dimensional representations [10–14] have been studied. When Kac obtained his classification [4] of simple Lie superalgebras, he subdivided them into the classical Lie superalgebras and the Lie superalgebras of Cartan type. The classical Lie superalgebras consist of the basic Lie superalgebras—$A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$ and the exceptional ones $D(2, 1; \alpha)$, $G(3)$ and $F(4)$—and the strange series $P(n)$ and $Q(n)$. The basic Lie superalgebras have made their appearance in various physical models. As far as we know, the strange Lie superalgebras have not been used in relation to any physical model or example. In this paper, we shall discuss the strange Lie superalgebra $Q(1)$ of rank 1; more precisely, we shall be dealing with its central extension which is usually denoted by $q(2)$ [15]. It will be shown that $q(2)$ has a class of interesting representations V_p labelled by a positive integer p. These representations allow for certain realizations of $q(2)$, and it will be shown that these realizations, in turn, are appropriate for the study of certain physical models: the so-called sphaleron model, the Moszkowski model, and the Jaynes–Cummings model.

The strange Lie superalgebras $q(n)$ can be considered as a super-analogue of $gl(n)$. Representations of $q(n)$ have been studied from the mathematical point of view. In [15–17], the finite dimensional irreducible *graded* representations of $q(n)$ have been determined together with their characters, both in the so-called typical and atypical cases. These representations possess the strange property that the multiplicity of the highest weight is in general greater than 1 [16]. More recently, a new class of finite dimensional irreducible representations of $q(n)$ was determined [18]. These representations are *not graded* and thus they are not among the ones classified by Penkov and Serganova [16]. However, they possess many other interesting properties: the highest weight has multiplicity 1, they can be equipped with an inner product, and in an apropriate context they can be considered as Fock spaces.

In the present paper we shall concentrate on these representations for the Lie superalgebra $q(2)$. The representations V_p are of dimension $2p$ (p is a positive integer). When decomposed to the even subalgebra $gl(2)$ of $q(2)$, V_p consists of the direct sum of two $gl(2)$ irreps: one of dimension $p + 1$ and the other of dimension $p - 1$. Having two $gl(2)$ irreps of such dimensions as part of an irreducible representation of another algebra (namely $q(2)$), will help in determining physical applications for the representations V_p.

The structure of this paper is as follows. In section 2, the algebra $q(2)$ and its class of representations V_p are defined. In section 3, we shall discuss a relation between these representations and certain representations of $so(4)$. Two realizations of $q(2)$ and of the corresponding representations V_p will be stated in section 4. The appearance and usefullness of these realizations in physical models will then be illustrated in the following sections: the sphaleron model in section 5, the Moszkowski model in section 6 and the Jaynes–Cummings model in section 7.

2. The Lie superalgebra $q(2)$ and the representations V_p

For the definition of $q(n)$ and a corresponding class of representations, we refer to [18]. Here we shall deal only with the case $n = 2$. The Lie superalgebra $q(2)$ has a basis consisting of four even elements $e_{ij}^{\bar{0}}$ $(i, j = 0, 1)$ and four odd elements $e_{ij}^{\bar{1}}$ $(i, j = 0, 1)$, satisfying the bracket relation

$$[\![e_{ij}^{\sigma}, e_{kl}^{\theta}]\!] = \delta_{jk} e_{il}^{\sigma+\theta} - (-1)^{\sigma\theta} \delta_{il} e_{kj}^{\sigma+\theta} \tag{1}$$

where $\sigma, \theta \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, and $i, j, k, l \in \{0, 1\}$. Here, $[\![\ ,\ ]\!]$ stands for the Lie superalgebra bracket, which could be a commutator or an anti-commutator, depending on the grading of the elements considered. We write explicitly $[\ ,\ ]$ (resp. $\{\ ,\ \}$) if this stands for a commutator (resp. anti-commutator).

It is clear that the even part of $q(2)$ (i.e. the four elements with upper index equal to $\bar{0}$) is the Lie algebra $gl(2)$. For convenience, a different notation will be introduced for the root vectors, i.e. the elements e_{ij}^{σ} with $i \neq j$, since these elements can be interpreted as 'creation and annihilation operators' for $q(2)$ [18]. So we put:

$$b^{+} = e_{10}^{\bar{0}} \qquad b^{-} = e_{01}^{\bar{0}} \tag{2}$$

$$f^{+} = e_{10}^{\bar{1}} \qquad f^{-} = e_{01}^{\bar{1}}. \tag{3}$$

These operators satisfy certain triple relations (see [18, equations (8)–(11)]), and together with their supercommutators they form a basis of $q(2)$.

The algebra $q(2)$ has finite dimensional representations labelled by a positive integer p. The representation space V_p arises as a quotient module $V_p = \overline{V}_p/M_p$ of an infinite dimensional $q(2)$ module $\overline{V}_p$ by its maximal submodule M_p [18]. The space $\overline{V}_p$ is spanned by the vectors

$$\begin{aligned} v_k &= (b^+)^k v_0 \qquad && k = 0, 1, \ldots \\ w_k &= (b^+)^{k-1} f^+ v_0 \qquad && k = 1, 2, \ldots \end{aligned} \tag{4}$$

where v_0 is a vacuum (or highest weight vector) satisfying:

$$\begin{aligned} &e_{00}^{\bar{0}} v_0 = p v_0 \qquad e_{00}^{\bar{1}} v_0 = \sqrt{p} v_0 \\ &e_{11}^{\bar{0}} v_0 = 0 \qquad e_{11}^{\bar{1}} v_0 = 0 \\ &b^- v_0 = f^- v_0 = 0. \end{aligned} \tag{5}$$

The following actions in $\overline{V}_p$ of the creation and annihilation operators on v_k and w_k can be computed:

$$\begin{aligned} &b^+ v_k = v_{k+1} \qquad b^+ w_k = w_{k+1} \\ &f^+ v_k = w_{k+1} \qquad f^+ w_k = 0 \\ &b^- v_k = k(p - k + 1) v_{k-1} \\ &f^- v_k = k\sqrt{p}\, v_{k-1} - k(k-1) w_{k-1} \\ &b^- w_k = \sqrt{p}\, v_{k-1} + (k-1)(p-k) w_{k-1} \\ &f^- w_k = p v_{k-1} - (k-1)\sqrt{p}\, w_{k-1}. \end{aligned} \tag{6}$$

In $\overline{V}_p$, $v_p - \sqrt{p}\, w_p$ is a primitive vector (the actions of b^- and f^- on it are zero) generating the submodule M_p. The quotient module $V_p = \overline{V}_p/M_p$ is therefore a finite dimensional module. A set of basis vectors of V_p, together with the corresponding weight in the natural basis (ϵ_0, ϵ_1) of the $gl(2)$ weight space, is given by

$$\begin{array}{ll} v_0 & p\epsilon_0 \\ v_1, w_1 & (p-1)\epsilon_0 + \epsilon_1 \\ v_2, w_2 & (p-2)\epsilon_0 + 2\epsilon_1 \\ \vdots & \vdots \\ v_{p-1}, w_{p-1} & \epsilon_0 + (p-1)\epsilon_1 \\ v_p + \sqrt{p}\, w_p & p\epsilon_1. \end{array} \tag{7}$$

The top and bottom weights have multiplicity 1, the other weights have multiplicity 2. Observe that we use the same notation for the vectors in V_p and $\overline{V}_p$.

From the above weight structure one can determine the decomposition of this finite dimensional $q(2)$ module with respect to the even subalgebra $gl(2) \subset q(2)$:

$$V_p \to (p, 0) \oplus (p-1, 1) \qquad (p > 1). \tag{8}$$

So V_p splits into two irreducible $gl(2)$ modules, both of which have been labelled by their highest weights (in the (ϵ_0, ϵ_1)-basis). In other words, the two components of the $gl(2)$ representations have dimensions $p + 1$ and $p - 1$; often this $gl(2)$ representation would be denoted by $\mathcal{D}^{(\frac{p}{2})} \oplus \mathcal{D}^{(\frac{p}{2}-1)}$.

The actions of the remaining $q(2)$ basis elements on the representation space V_p can easily be determined:

$$\begin{aligned} &e^{\bar{0}}_{00} v_k = (p-k)v_k && e^{\bar{0}}_{00} w_k = (p-k)w_k \\ &e^{\bar{0}}_{11} v_k = k v_k && e^{\bar{0}}_{11} w_k = k w_k \\ &e^{\bar{1}}_{00} v_k = \sqrt{p}v_k - k w_k && e^{\bar{1}}_{00} w_k = v_k - \sqrt{p} w_k \\ &e^{\bar{1}}_{11} v_k = k w_k && e^{\bar{1}}_{11} w_k = v_k. \end{aligned} \tag{9}$$

On the representation space V_p, a positive-definite metric can be introduced by requiring

$$\langle v_0|v_0\rangle = 1 \qquad \langle b^+ v|v'\rangle = \langle v|b^- v'\rangle \qquad \langle f^+ v|v'\rangle = \langle v|f^- v'\rangle \qquad \forall v, v' \in V_p. \tag{10}$$

Then

$$\begin{aligned} \langle v_k|v_l\rangle &= \delta_{kl}\frac{k!\,p!}{(p-k)!} \\ \langle w_k|w_l\rangle &= \delta_{kl}\frac{(k-1)!\,p!}{(p-k)!} \\ \langle v_k|w_l\rangle &= \delta_{kl}\frac{k!\,p!}{(p-k)!\sqrt{p}}. \end{aligned} \tag{11}$$

Because of the last relation, the basis (7) is not orthogonal with respect to this metric, so it will be convenient to introduce another (and more convenient) orthogonal basis of V_p as follows:

$$\Lambda_k = \frac{(p-k)!}{p!}\, v_k \qquad (k = 0, 1, \ldots, p-1) \tag{12}$$

$$\Lambda_p = \frac{1}{2p!}(v_p + \sqrt{p}\, w_p) \tag{13}$$

$$\chi_l = \frac{(p-l-1)!}{p!}(v_l - \sqrt{p}\, w_l) \qquad (l = 1, 2, \ldots, p-1). \tag{14}$$

The action of the creation and annihilation operators on this basis reads (in the following equations, $k = 0, 1, \ldots, p$ and $l = 1, 2, \ldots, p-1$)

$$\begin{aligned} &b^-\Lambda_k = k\Lambda_{k-1} \\ &b^-\chi_l = (l-1)\chi_{l-1} \\ &b^+\Lambda_k = (p-k)\Lambda_{k+1} \\ &b^+\chi_l = (p-l-1)\chi_{l+1} \\ &f^-\Lambda_k = (k\Lambda_{k-1} + k(k-1)\chi_{k-1})/\sqrt{p} \\ &f^-\chi_l = -(\Lambda_{l-1} + (l-1)\chi_{l-1})/\sqrt{p} \\ &f^+\Lambda_k = ((p-k)\Lambda_{k+1} - (p-k)(p-k-1)\chi_{k+1})/\sqrt{p} \\ &f^+\chi_l = (\Lambda_{l+1} - (p-l-1)\chi_{l+1})/\sqrt{p}. \end{aligned} \tag{15}$$

Note that in all the computations, one has to work in the quotient module $V_p = \overline{V}_p/M_p$, where M_p is generated by the primitive vector $v_p - \sqrt{p}w_p$ of $\overline{V}_p$. This often requires a separate calculation for the cases $k = p$ or $k = p-1$. For example,

$$b^+\Lambda_{p-1} = \frac{1}{p!}v_p = \frac{1}{p!}\left(v_p - \frac{1}{2}(v_p - \sqrt{p}w_p)\right) = \frac{1}{2p!}(v_p + \sqrt{p}w_p) = \Lambda_p.$$

The actions of the remaining $q(2)$ elements on this basis are given by

$$\begin{aligned}
e_{00}^{\bar{0}}\Lambda_k &= (p-k)\Lambda_k \\
e_{00}^{\bar{0}}\chi_l &= (p-l)\chi_l \\
e_{11}^{\bar{0}}\Lambda_k &= k\,\Lambda_k \\
e_{11}^{\bar{0}}\chi_l &= l\,\chi_l \\
e_{00}^{\bar{1}}\Lambda_k &= ((p-k)\Lambda_k + k(p-k)\chi_k)/\sqrt{p} \\
e_{00}^{\bar{1}}\chi_l &= (\Lambda_l - (p-l)\chi_l)/\sqrt{p} \\
e_{11}^{\bar{1}}\Lambda_k &= (k\Lambda_k - k(p-k)\chi_k)/\sqrt{p} \\
e_{11}^{\bar{1}}\chi_l &= -(\Lambda_l + l\chi_l)/\sqrt{p}
\end{aligned} \tag{16}$$

where again $k = 0, 1, \ldots, p$ and $l = 1, 2, \ldots, p-1$. Observe that the subalgebra $gl(2)$ with basis $\left\{b^+, b^-, e_{00}^{\bar{0}}, e_{11}^{\bar{0}}\right\}$ acts irreducibly on the vectors Λ_k ($k = 0, 1, \ldots, p$) and χ_l ($l = 1, 2, \ldots, p-1$); so from here the decomposition of V_p into two irreducible $gl(2)$ irreps is obvious.

3. A relation with $so(4)$ representations

Consider the Lie algebra $so(4) \equiv sl(2) \oplus sl(2)$ with generators J_i and K_i ($i = 0, \pm$) and commutation relations

$$\begin{aligned}
&[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_0 \\
&[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = K_0 \\
&[J_i, K_j] = 0.
\end{aligned} \tag{17}$$

Rather than dealing with the abstract generators of $so(4)$, we shall consider these generators in a particular representation. The operators J_i ($i = 0, \pm$) are realized in the representation $\mathcal{D}^{(\frac{p-1}{2})}$ of $sl(2)$ (with p a positive integer), and the operators K_i ($i = 0, \pm$) are realized in the representation $\mathcal{D}^{(\frac{1}{2})}$ of $sl(2)$. We shall continue to denote the representatives of the abstract operators (17) by the same names, J_i and K_i. Thus the operators K_i satisfy

$$(K_\pm)^2 = 0 \qquad K_0^2 = \tfrac{1}{4}I \qquad \{K_+, K_-\} = I \qquad \{K_0, K_\pm\} = 0 \tag{18}$$

where I is the identity operator.

The Lie algebra $so(4) = sl(2) \oplus sl(2)$ has the subalgebra $sl(2)$ with generators $J_i + K_i$ ($i = 0, \pm$). Since in the present realization the tensor product $\mathcal{D}^{(\frac{p-1}{2})} \otimes \mathcal{D}^{(\frac{1}{2})}$ decomposes as $\mathcal{D}^{(\frac{p}{2})} \oplus \mathcal{D}^{(\frac{p}{2}-1)}$, the representation of $so(4)$ considered here decomposes as $\mathcal{D}^{(\frac{p}{2})} \oplus \mathcal{D}^{(\frac{p}{2}-1)}$ with respect to this $sl(2)$ subalgebra. This implies that the $so(4)$ representation space is isomorphic to the space V_p, with the same $sl(2)$ action. Denoting the representatives of $q(2)$ in V_p again by $b^\pm, f^\pm, e_{ii}^\sigma$ ($\sigma = \bar{0}, \bar{1}$, $i = 0, 1$), the following identification holds:

$$\begin{aligned}
&b^- = J_+ + K_+ \qquad b^+ = J_- + K_- \qquad e_{00}^{\bar{0}} - e_{11}^{\bar{0}} = 2J_0 + 2K_0 \\
&f^- = \sqrt{p}K_+ \qquad f^+ = \sqrt{p}K_- \qquad e_{00}^{\bar{1}} - e_{11}^{\bar{1}} = 2\sqrt{p}K_0 \\
&e_{00}^{\bar{0}} + e_{11}^{\bar{0}} = pI \qquad e_{00}^{\bar{1}} + e_{11}^{\bar{1}} = \frac{2}{\sqrt{p}}\left(2J_0K_0 + J_+K_- + J_-K_+ + \frac{1}{2}\right).
\end{aligned} \tag{19}$$

These relations can be verified by considering the representations of the $so(4)$ generators in a standard basis of $\mathcal{D}(\frac{p-1}{2}, \frac{1}{2}) = \mathcal{D}^{(\frac{p-1}{2})} \otimes \mathcal{D}^{(\frac{1}{2})}$, and comparing with (15)–(16). Indeed, let the standard basis of $\mathcal{D}(\frac{p-1}{2}, \frac{1}{2})$ be given by

$$\left|\frac{p-1}{2}, m\right\rangle \otimes \left|\frac{1}{2}, \mu\right\rangle$$

where $m = -\frac{p-1}{2}, -\frac{p-1}{2} + 1, \ldots, \frac{p-1}{2}$ and $\mu = \pm\frac{1}{2}$, then the standard action of the $so(4)$ basis elements reads

$$\begin{aligned}
J_0 \left|\frac{p-1}{2}, m\right\rangle \otimes \left|\frac{1}{2}, \mu\right\rangle &= m \left|\frac{p-1}{2}, m\right\rangle \otimes \left|\frac{1}{2}, \mu\right\rangle \\
J_\pm \left|\frac{p-1}{2}, m\right\rangle \otimes \left|\frac{1}{2}, \mu\right\rangle &= \left(\left(\frac{p-1}{2} \mp m\right)\left(\frac{p-1}{2} \pm m + 1\right)\right)^{1/2} \\
&\quad \times \left|\frac{p-1}{2}, m \pm 1\right\rangle \otimes \left|\frac{1}{2}, \mu\right\rangle \\
K_0 \left|\frac{p-1}{2}, m\right\rangle \otimes \left|\frac{1}{2}, \mu\right\rangle &= \mu \left|\frac{p-1}{2}, m\right\rangle \otimes \left|\frac{1}{2}, \mu\right\rangle \\
K_\pm \left|\frac{p-1}{2}, m\right\rangle \otimes \left|\frac{1}{2}, \mu\right\rangle &= \left(\left(\frac{1}{2} \mp \mu\right)\left(\frac{1}{2} \pm \mu + 1\right)\right)^{1/2} \\
&\quad \times \left|\frac{p-1}{2}, m\right\rangle \otimes \left|\frac{1}{2}, \mu \pm 1\right\rangle .
\end{aligned} \tag{20}$$

Using the following relation between the (Λ_k, χ_l)-basis and the present one,

$$\begin{aligned}
\Lambda_k &= \sqrt{\frac{(p-k)!k!}{p!}} \left(\sqrt{\frac{p-k}{p}} \left|\frac{p-1}{2}, \frac{p-1}{2} - k\right\rangle \otimes \left|\frac{1}{2}, \frac{1}{2}\right\rangle \right. \\
&\quad \left. + \sqrt{\frac{k}{p}} \left|\frac{p-1}{2}, \frac{p+1}{2} - k\right\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right) \\
\chi_l &= \sqrt{\frac{(p-l-1)!(l-1)!}{p!}} \left(\sqrt{\frac{l}{p}} \left|\frac{p-1}{2}, \frac{p-1}{2} - l\right\rangle \otimes \left|\frac{1}{2}, \frac{1}{2}\right\rangle \right. \\
&\quad \left. - \sqrt{\frac{p-l}{p}} \left|\frac{p-1}{2}, \frac{p+1}{2} - l\right\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right)
\end{aligned}$$

it is straightforward to verify that (19) holds, using the actions (15)–(16) and (20).

Observe that $so(4)$ has two Casimir operators C_1 and C_2, which are independent in general:

$$C_1 = J_0^2 + K_0^2 + \tfrac{1}{2}\{J_+, J_-\} + \tfrac{1}{2}\{K_+, K_-\} \tag{21}$$

$$C_2 = J_0^2 - K_0^2 + \tfrac{1}{2}\{J_+, J_-\} - \tfrac{1}{2}\{K_+, K_-\}. \tag{22}$$

In the present representation, however, these operators are not independent. They can be rewritten in terms of the $q(2)$ operators, in which case C_1 and C_2 coincide apart from a multiple of the operator $e_{00}^{\bar{0}} + e_{11}^{\bar{0}}$ (with eigenvalue p in the representation). The Casimirs C_1 and C_2 have the value $2p^2 - 1$ and $2p^2 - 4$, respectively.

4. Two realizations of $q(2)$ and its representation V_p

In order to find applications of the algebra $q(2)$ and its representations V_p, it will be useful to construct certain differential realizations of $q(2)$. Here we shall give two different differential realizations. The main difference comes from the distinction between the spaces of polynomials that the $q(2)$ elements act upon.

A simple realization of $q(2)$ is found by realizing the basis elements Λ_k, χ_l as follows:

$$\Lambda_k = \begin{pmatrix} x^k \\ 0 \end{pmatrix} \quad k = 0, 1, \ldots, p \qquad \chi_l = \begin{pmatrix} 0 \\ x^{l-1} \end{pmatrix} \quad l = 1, 2, \ldots, p-1. \tag{23}$$

Thus the basis elements are (2×1)-arrays of polynomials in a variable x. The representation space can then be identified with

$$\begin{pmatrix} \mathcal{P}(p) \\ \mathcal{P}(p-2) \end{pmatrix} \tag{24}$$

where $\mathcal{P}(m)$ stands for the space of polynomials in x of degree at most m, thus $\mathcal{P}(m)$ has a basis $\{1, x, \ldots, x^m\}$. The Lie superalgebra $q(2)$ will have a realization preserving the space (24).

With this realization of the basis vectors Λ_k and χ_l, a differential realization for $q(2)$ is easily derived from (15)–(16). There comes

$$\begin{aligned}
b^- &= \frac{\mathrm{d}}{\mathrm{d}x} \qquad b^+ = -x^2\frac{\mathrm{d}}{\mathrm{d}x} + (p-1)x + x\sigma_3 \\
e_{00}^{\bar{0}} - e_{11}^{\bar{0}} &= -2x\frac{\mathrm{d}}{\mathrm{d}x} + p - 1 + \sigma_3 \qquad e_{00}^{\bar{0}} + e_{11}^{\bar{0}} = p \\
f^- &= \frac{1}{\sqrt{p}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\sigma_3 - \sigma_+ + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma_-\right) \\
f^+ &= \frac{1}{\sqrt{p}}\left(-x^2\frac{\mathrm{d}}{\mathrm{d}x} + (p-1)x\right)\sigma_3 + \frac{1}{\sqrt{p}}x + \frac{1}{\sqrt{p}}x^2\sigma_+ \\
&\quad - \frac{1}{\sqrt{p}}\left(x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2(1-p)x\frac{\mathrm{d}}{\mathrm{d}x} + p(p-1)\right)\sigma_- \\
e_{00}^{\bar{1}} - e_{11}^{\bar{1}} &= \frac{1}{\sqrt{p}}\left(-2x\frac{\mathrm{d}}{\mathrm{d}x} + p - 1\right)\sigma_3 + \frac{1}{\sqrt{p}} + \frac{2}{\sqrt{p}}x\sigma_+ \\
&\quad + \frac{2}{\sqrt{p}}\left(-x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (p-1)\frac{\mathrm{d}}{\mathrm{d}x}\right)\sigma_- \\
e_{00}^{\bar{1}} + e_{11}^{\bar{1}} &= \sqrt{p}\sigma_3.
\end{aligned} \tag{25}$$

Herein, $\sigma_\pm$ and σ_3 are the common notations for the Pauli matrices. We shall refer to (25) as the first differential realization of $q(2)$.

A second useful realization of $q(2)$ will be found by considering a different basis for V_p. Let, for $k = 0, 1, \ldots, p-1$,

$$\mu_k = \Lambda_{p-k} - k\chi_{p-k} \tag{26}$$

$$\mu_{p+k} = \Lambda_{p-k-1} + (p-k-1)\chi_{p-k-1}. \tag{27}$$

Then the action of the $q(2)$ operators on this new basis reads

$$
\begin{aligned}
&b^+\mu_k = k\mu_{k-1} \qquad b^+\mu_{p+k} = \mu_k + k\mu_{p+k-1}\\
&f^+\mu_k = 0 \qquad f^+\mu_{p+k} = \sqrt{p}\mu_k\\
&b^-\mu_k = (p-k-1)\mu_{k+1} + \mu_{p+k} \qquad b^-\mu_{p+k} = (p-k-1)\mu_{p+k+1}\\
&f^-\mu_k = \sqrt{p}\mu_{p+k} \qquad f^-\mu_{p+k} = 0\\
&\left(e_{00}^{\bar 0} + e_{11}^{\bar 0}\right)\mu_k = p\mu_k \qquad \left(e_{00}^{\bar 0} + e_{11}^{\bar 0}\right)\mu_{p+k} = p\mu_{p+k}\\
&\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right)\mu_k = (2k-p)\mu_k \qquad \left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right)\mu_{p+k} = (2k+2-p)\mu_{p+k}\\
&\left(e_{00}^{\bar 1} + e_{11}^{\bar 1}\right)\mu_k = \frac{1}{\sqrt{p}}(p-2k)\mu_k + \frac{1}{\sqrt{p}}(2k)\mu_{p+k-1}\\
&\left(e_{00}^{\bar 1} + e_{11}^{\bar 1}\right)\mu_{p+k} = \frac{1}{\sqrt{p}}(2k+2-p)\mu_{p+k} + \frac{2}{\sqrt{p}}(p-k-1)\mu_{k+1}\\
&\left(e_{00}^{\bar 1} - e_{11}^{\bar 1}\right)\mu_k = -\sqrt{p}\mu_k \qquad \left(e_{00}^{\bar 1} - e_{11}^{\bar 1}\right)\mu_{p+k} = \sqrt{p}\mu_{p+k}.
\end{aligned}
\tag{28}
$$

Just as the basis Λ_k, χ_l could be represented by (2×1)-arrays of polynomials in a variable, the same holds for the present basis. Let us consider

$$
\mu_k = \begin{pmatrix} x^k \\ 0 \end{pmatrix} \qquad \mu_{p+k} = \begin{pmatrix} 0 \\ x^k \end{pmatrix} \qquad k = 0, 1, \ldots, p-1. \tag{29}
$$

When expressed in this basis, the Lie superalgebra will have a realization preserving the space

$$
\begin{pmatrix} \mathcal{P}(p-1) \\ \mathcal{P}(p-1) \end{pmatrix}. \tag{30}
$$

Following from the action given in (28), this realization reads

$$
\begin{aligned}
&b^- = -x^2\frac{\mathrm{d}}{\mathrm{d}x} + (p-1)x + \sigma_- \qquad b^+ = \frac{\mathrm{d}}{\mathrm{d}x} + \sigma_+\\
&e_{00}^{\bar 0} - e_{11}^{\bar 0} = 2x\frac{\mathrm{d}}{\mathrm{d}x} + 1 - p - \sigma_3 \qquad e_{00}^{\bar 0} + e_{11}^{\bar 0} = p\\
&f^- = \sqrt{p}\sigma_- \qquad f^+ = \sqrt{p}\sigma_+ \qquad e_{00}^{\bar 1} - e_{11}^{\bar 1} = -\sqrt{p}\sigma_3\\
&e_{00}^{\bar 1} + e_{11}^{\bar 1} = \frac{1}{\sqrt{p}}\left(-2x\frac{\mathrm{d}}{\mathrm{d}x}\sigma_3 + 1 + (p-1)\sigma_3 + 2\frac{\mathrm{d}}{\mathrm{d}x}\sigma_- + 2(p-1)x\sigma_+ - 2x^2\frac{\mathrm{d}}{\mathrm{d}x}\sigma_+\right)
\end{aligned}
\tag{31}
$$

and will be referred to as the second differential realization of $q(2)$.

5. Sphaleron model

In this section, we discuss a (physical) system of two coupled equations. In particular, this system will have algebraic solutions in the representation spaces (24) and (30). Such a system arises in the study of the stability of sphalerons [19] (i.e. unstable classical solutions) in the Abelian gauge–Higgs model in 1 + 1 dimensions. The relevant equations read [20]

$$
\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \lambda - \theta^2 k^2\, sn^2\right) f(y) - 2\theta k\; cn\; dn\; g(y) = 0 \tag{32}
$$

$$
\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \lambda + 1 + k^2 - (\theta^2+2)k^2\, sn^2\right) g(y) - 2\theta k\; cn\; dn\; f(y) = 0 \tag{33}
$$

and are considered on the Hilbert space of periodic functions over $[0, 4K(k)]$ ($K(k)$ is the complete elliptic integral of the second type). The three elliptic functions [21] $sn = sn(y,k)$,

$cn = cn(y,k)$ and $dn = dn(y,k)$ are periodic with respective periods $4K(k)$, $4K(k)$ and $2K(k)$. The spectral parameter λ is the mode eigenvalue of the system while θ stands for the mass ratio $2M_{\mathrm{H}}/M_{\mathrm{W}}$, M_{H} and M_{W} being respectively the masses of the Higgs and gauge bosons.

Introducing the following new function

$$W(y) \equiv \frac{\mathrm{d}f(y)}{\mathrm{d}y} - \theta k\, sn\; g(y) \tag{34}$$

as well as of the change of variables

$$x = sn^2(y,k) \tag{35}$$

the system (32)–(33) becomes

$$\left(4x(1-x)(1-k^2x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2(1-2(1+k^2)x+3k^2x^2)\frac{\mathrm{d}}{\mathrm{d}x} + \lambda - k^2\theta^2 x\right)W(x) = 0 \tag{36}$$

$$\left(4x(1-x)(1-k^2x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2(-1+k^2x^2)\frac{\mathrm{d}}{\mathrm{d}x} + \lambda - k^2\theta^2 x\right)f(x) = -2\sqrt{\frac{(1-x)(1-k^2x)}{x}}W(x). \tag{37}$$

It has been proved [20] that this system has algebraic solutions in a $2p$-dimensional space if

$$\theta^2 = 2p(2p+1) \quad \text{or} \quad \theta^2 = 2p(2p-1). \tag{38}$$

This result suggests a connection between this sphaleron model and the $q(2)$-representations we are dealing with. More precisely, if $\theta^2 = 2p(2p+1)$, we can put either

$$W(x) = P_{p-1}(x) + xQ_{p-1}(x) \qquad f(x) = \sqrt{x(1-x)(1-k^2x)}P_{p-1}(x) \tag{39}$$

where $P_m(x)$ and $Q_m(x)$ stand for polynomials of degree m in x, or else

$$W(x) = \sqrt{(1-x)(1-k^2x)}P_{p-1}(x) \qquad f(x) = \sqrt{x}(P_{p-1}(x) + xQ_{p-1}(x)). \tag{40}$$

Under one of these two substitutions, the system of equations (40)–(41) has polynomial solutions for $P_{p-1}(x)$ and $Q_{p-1}(x)$. Indeed, in the case (39), the system of equations becomes

$$\left(4x(1-x)(1-k^2x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2(1-4(1+k^2)x+7k^2x^2)\frac{\mathrm{d}}{\mathrm{d}x} + \lambda - k^2(4p^2+2p-6)x\right)P_{p-1}(x) = -2Q_{p-1}(x) \tag{41}$$

$$\left(4x(1-x)(1-k^2x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2(5-6(1+k^2)x+7k^2x^2)\frac{\mathrm{d}}{\mathrm{d}x} + \lambda - 4(1+k^2) - k^2(4p^2+2p-6)x\right)Q_{p-1}(x) = \left((8k^2x-4(1+k^2))\frac{\mathrm{d}}{\mathrm{d}x} - 6k^2\right)P_{p-1}(x). \tag{42}$$

The differential operators of (41)–(42) map any element $\begin{pmatrix} P_{p-1}(x) \\ Q_{p-1}(x) \end{pmatrix}$ of the space (30) into an element of the same space. Thus (41)–(42) reduces to an algebraic eigenvalue system for λ.

The differential operator can be written as

$$\Delta_{(39)} + \lambda = 4x\frac{d^2}{dx^2} - 4(1+k^2)x^2\frac{d^2}{dx^2} + 4k^2x^3\frac{d^2}{dx^2} + (6 - 10(1+k^2)x + 14k^2x^2)\frac{d}{dx} + (-4 + 2(1+k^2)x)\frac{d}{dx}\sigma_3 + (-4p^2 - 2p + 6)k^2x - 2(1+k^2) + 2(1+k^2)\sigma_3 + 2\sigma_+ - 6k^2\sigma_- + (4(1+k^2) - 8k^2x)\frac{d}{dx}\sigma_- + \lambda. \quad (43)$$

Since this operator leaves the space of polynomials (30) invariant, we might expect that it can be expressed in terms of the $q(2)$-generators realized as in the so-called second realization (i.e. as in (31)). We actually have

$$\begin{aligned}\Delta_{(39)} + \lambda = {}& 2\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right) b^+ - \frac{2}{\sqrt{p}}\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right) f^+ - 2k^2\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right) b^- \\ & + \frac{2k^2}{\sqrt{p}}\left(e^{\bar 1}_{00} - e^{\bar 1}_{11}\right) b^- - (1+k^2)\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right)^2 + \frac{2}{\sqrt{p}} b^+\left(e^{\bar 1}_{00} - e^{\bar 1}_{11}\right) \\ & - \frac{6}{p} f^+\left(e^{\bar 1}_{00} - e^{\bar 1}_{11}\right) + (1+k^2)\frac{1}{\sqrt{p}}\left(e^{\bar 1}_{00} - e^{\bar 1}_{11}\right)\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right) \\ & - \frac{2k^2}{\sqrt{p}}\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right) f^- + \frac{2k^2}{p}\left(e^{\bar 1}_{00} - e^{\bar 1}_{11}\right) f^- + 4(1+k^2)\frac{1}{\sqrt{p}} b^+ f^- \\ & - 4(1+k^2)\frac{1}{p} f^+ f^- - 2k^2(1-p)\frac{1}{\sqrt{p}} f^- + 2(p+2)b^+ \\ & - 2(p-1)\frac{1}{\sqrt{p}} f^+ - 6k^2 p b^- - (1+k^2)(2p+1)\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right) \\ & + (1+k^2)\sqrt{p}\left(e^{\bar 1}_{00} - e^{\bar 1}_{11}\right) - p(p+1)(1+k^2) + \lambda. \end{aligned} \quad (44)$$

The same result holds for the case (44) where we obtain

$$\begin{aligned}\Delta_{(40)} + \lambda = {}& 2\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right) b^+ - \frac{2}{\sqrt{p}}\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right) f^+ - 2k^2\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right) b^- \\ & + \frac{2k^2}{\sqrt{p}}\left(e^{\bar 1}_{00} - e^{\bar 1}_{11}\right) b^- - (1+k^2)\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right)^2 + \frac{2}{\sqrt{p}} b^+\left(e^{\bar 1}_{00} - e^{\bar 1}_{11}\right) \\ & - \frac{6}{p} f^+\left(e^{\bar 1}_{00} - e^{\bar 1}_{11}\right) + (1+k^2)\frac{1}{\sqrt{p}}\left(e^{\bar 1}_{00} - e^{\bar 1}_{11}\right)\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right) \\ & - \frac{2k^2}{\sqrt{p}}\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right) f^- + \frac{2k^2}{p}\left(e^{\bar 1}_{00} - e^{\bar 1}_{11}\right) f^- + 4(1+k^2)\frac{1}{\sqrt{p}} b^+ f^- \\ & - 4(1+k^2)\frac{1}{p} f^+ f^- + 2k^2\sqrt{p} f^- + 2(p+2)b^+ - 2\sqrt{p} f^+ \\ & - 6k^2 p b^- - (1+k^2)(2p+1)\left(e^{\bar 0}_{00} - e^{\bar 0}_{11}\right) \\ & + (1+k^2)\frac{1}{\sqrt{p}}(p+1)\left(e^{\bar 1}_{00} - e^{\bar 1}_{11}\right) - p(p+1)(1+k^2) + \lambda. \end{aligned} \quad (45)$$

In the case that $\theta^2 = 2p(2p-1)$, we can consider either

$$W(x) = \sqrt{x}\,Q_{p-1}(x) \qquad f(x) = \sqrt{(1-x)(1-k^2x)}\,P_{p-1}(x) \quad (46)$$

or else

$$W(x) = \sqrt{x(1-x)(1-k^2x)}\,Q_{p-2}(x) \qquad f(x) = P_p(x). \quad (47)$$

With the substitution (46), the space preserved by the differential operator is still (30). Acting on an array of polynomials $\begin{pmatrix} P_{p-1}(x) \\ Q_{p-1}(x) \end{pmatrix}$, the equation reduces to an algebraic eigenvalue equation; using the second realization (31) one is again able to express the differential operator subtended by this physical model in terms of the $q(2)$-generators. Explicitly this reads:

$$\begin{aligned}
\Delta_{(46)} + \lambda = {} & 2\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right) b^+ - \frac{2}{\sqrt{p}}\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right) f^+ - 2k^2\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right) b^- \\
& + \frac{2k^2}{\sqrt{p}}\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right) f^- + \frac{2k^2}{\sqrt{p}}\left(e_{00}^{\bar 1} - e_{11}^{\bar 1}\right) b^- - \frac{2k^2}{p}\left(e_{00}^{\bar 1} - e_{11}^{\bar 1}\right) f^- \\
& - (1+k^2)\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right)^2 + \frac{2}{\sqrt{p}} b^+\left(e_{00}^{\bar 1} - e_{11}^{\bar 1}\right) - \frac{6}{p} f^+\left(e_{00}^{\bar 1} - e_{11}^{\bar 1}\right) \\
& + (1+k^2)\frac{1}{\sqrt{p}}\left(e_{00}^{\bar 1} - e_{11}^{\bar 1}\right)\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right) + 2pb^+ + \frac{2}{\sqrt{p}}(3-p) f^+ \\
& - 2k^2(3p-2)b^- + 2k^2(3p-2)\frac{1}{\sqrt{p}} f^- + (1+k^2)(-2p+1)\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right) \\
& - (1+k^2)(1-p)\frac{1}{\sqrt{p}}\left(e_{00}^{\bar 1} - e_{11}^{\bar 1}\right) - p(p-1)(1+k^2) + \lambda. \qquad (48)
\end{aligned}$$

The context for the substitution (47) is slightly different, so it deserves more attention. This time, the differential operator coming from the system (36)–(37) acts on an element $\begin{pmatrix} P_{p}(x) \\ Q_{p-2}(x) \end{pmatrix}$ from the space (24). Since also this space is a representation space for $q(2)$, as we have proved in the previous section, we can again expect that the differential operator can be written in terms of the $q(2)$-generators. This is indeed the case when using the first differential realization of $q(2)$ as given in (25). There comes

$$\begin{aligned}
\Delta_{(47)} + \lambda = {} & 2k^2 b^+\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right) - k^2 f^+\left(e_{00}^{\bar 1} + e_{11}^{\bar 1}\right) - \frac{1}{\sqrt{p}} b^-\left(e_{00}^{\bar 1} + e_{11}^{\bar 1}\right) \\
& - 2\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right) b^- + \frac{1}{\sqrt{p}} k^2 b^+\left(e_{00}^{\bar 1} + e_{11}^{\bar 1}\right) + 4(1+k^2) b^+ b^- \\
& + f^-\left(e_{00}^{\bar 1} + e_{11}^{\bar 1}\right) + \frac{1}{2}(1+k^2)\left(e_{00}^{\bar 1} - e_{11}^{\bar 1}\right)\left(e_{00}^{\bar 1} + e_{11}^{\bar 1}\right) \\
& - \frac{1}{2\sqrt{p}}(1+k^2)\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right)\left(e_{00}^{\bar 1} + e_{11}^{\bar 1}\right) + (2p-1)b^- + k^2\sqrt{p} f^+ \\
& + k^2(-6p+1)b^+ - \sqrt{p} f^- + (1+k^2)\left(2p + \frac{1}{2}\right)\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right) \\
& - \sqrt{p}(1+k^2)\left(e_{00}^{\bar 1} + e_{11}^{\bar 1}\right) - \frac{1}{2}\sqrt{p}(1+k^2)\left(e_{00}^{\bar 1} - e_{11}^{\bar 1}\right) \\
& + (-2p^2 + p)(1+k^2) + \lambda. \qquad (49)
\end{aligned}$$

We have thus written each of the differential operators $\Delta_{(39)}, \Delta_{(40)}, \Delta_{(46)}$ and $\Delta_{(47)}$ associated with the sphaleron model in terms of the $q(2)$ generators. The Lie superalgebra $q(2)$ acts as a 'spectrum generating superalgebra' for this physical model. More precisely both the sets of linear differential operators playing a role in the sphaleron model, those preserving the vector space of 2-arrays of polynomials of degrees $p-1$ and $p-1$ on the one hand and those preserving the vector space of 2-arrays of polynomials of degrees p and $p-2$ on the other hand, correspond to realizations of $q(2)$ and make the determination of λ possible. Such a determination is relatively straightforward due to the fact that the (linear) Lie superalgebra $q(2)$ has a particularly simple structure, much simpler than the algebras used in previous papers [20, 22] devoted to the calculation of λ. Indeed in these papers, the algebra $so(4)$ (for $\Delta_{(39)}$, $\Delta_{(40)}$ and $\Delta_{(46)}$) as well as an associative (non-linear) graded algebra denoted by $\mathcal{A}(2)$ (for $\Delta_{(47)}$) have been used for such a task and this required heavy techniques in connection with the study [22] of the irreps of this $\mathcal{A}(2)$. Such a simplification obtained by considering $q(2)$ instead of $\mathcal{A}(2)$ leads to the hope of a more direct diagonalization of the operators connected with $\mathcal{A}(n)$ [22] by using $q(n)$.

6. Moszkowski model

We now turn to the Moszkowski model [23]. This is a two-level model, each of the levels being N-fold degenerate with N_a particles of type a and N_b particles of type b. The state of each particle is specified by the quantum numbers $\sigma = \pm\frac{1}{2}$ (taking the value $\frac{1}{2}$ in the upper level and $-\frac{1}{2}$ in the lower level) and q which refers to the particular degenerate state within a given level. The corresponding Hamiltonian associated to the model reads [23]

$$H_M = c\,(J_0(a) - J_0(b)) + V\{\hat{J}_+, \hat{J}_-\} \tag{50}$$

where c is the energy difference between the two levels and V denotes the interaction strength. In (50), the operators $J_0(a)$, $J_\pm(a)$ are defined according to

$$J_0(a) = \frac{1}{2}\sum_q \left(a^+_{q,\frac{1}{2}} a^-_{q,\frac{1}{2}} - a^+_{q,-\frac{1}{2}} a^-_{q,-\frac{1}{2}} \right) \tag{51}$$

$$J_+(a) = \sum_q a^+_{q,\frac{1}{2}} a^-_{q,-\frac{1}{2}} \tag{52}$$

$$J_-(a) = \sum_q a^+_{q,-\frac{1}{2}} a^-_{q,\frac{1}{2}} \tag{53}$$

where $a^+_{q,\pm\frac{1}{2}}$ $(a^-_{q,\pm\frac{1}{2}})$ denotes the creation (annihilation) operator of a particle of type a in the state q with $\sigma = \pm\frac{1}{2}$. Similar definitions hold for $J_0(b)$, $J_\pm(b)$ and we also have

$$\hat{J}_i = J_i(a) + J_i(b) \qquad i = 0, \pm. \tag{54}$$

The operators $J_0(i)$, $J_\pm(i)$ $(i = a, b)$ satisfy the $so(4) \equiv sl(2) \oplus sl(2)$ commutation relations

$$[J_0(i), J_\pm(j)] = \pm\,\delta_{ij} J_\pm(i) \tag{55}$$

$$[J_+(i), J_-(j)] = 2\delta_{ij} J_0(i) \qquad (i, j = a, b). \tag{56}$$

Because of this $sl(2) \oplus sl(2)$ symmetry of the Moszkowski Hamiltonian, we can also expect the $q(2)$ Lie superalgebra to play a role within this model. Associating the operators J_i and K_i $(i = 0, \pm)$ of (17) with the current operators $J_i(b)$ and $J_i(a)$ respectively, we can rewrite H_M as

$$H_M = c\,(K_0 - J_0) + V(\{K_+, K_-\} + \{J_+, J_-\} + 2J_+K_- + 2J_-K_+). \tag{57}$$

According to (19), this can be rewritten as

$$H_M = c\left(\frac{1}{\sqrt{p}}\left(e^{\bar{1}}_{00} - e^{\bar{1}}_{11}\right) - \frac{1}{2}\left(e^{\bar{0}}_{00} - e^{\bar{0}}_{11}\right)\right) + V\left(\frac{1}{2}p^2 - \frac{1}{2}\left(e^{\bar{0}}_{00} - e^{\bar{0}}_{11}\right)^2 + \sqrt{p}\left(e^{\bar{1}}_{00} + e^{\bar{1}}_{11}\right)\right). \tag{58}$$

Although in principle the Hamiltonian H_M can be diagonalized using the expression (57) in terms of $so(4)$-generators, it turns out to be much simpler using the expression (58) in terms of $q(2)$-generators together with the second differential realization (35) of $q(2)$. Then the Hamiltonian becomes

$$H_M = c\left(-x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}(p-1) - \frac{1}{2}\sigma_3\right) + V\left(-2x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (2p-4)x\frac{\mathrm{d}}{\mathrm{d}x} + p + 2\frac{\mathrm{d}}{\mathrm{d}x}\sigma_- + 2(p-1)x\sigma_+ - 2x^2\frac{\mathrm{d}}{\mathrm{d}x}\sigma_+\right). \tag{59}$$

Considering the action of this on the representation space (34), or equivalently, the action (28) of (58) on the basis vectors (26)–(27), leads to an eigenvalue system that is almost trivial to solve, i.e.

$$E_0^+ = pV - \left(1 - \tfrac{p}{2}\right)c$$
$$E_k^\pm = -2Vk(k-p) + c\left(\tfrac{p}{2} - k\right) \pm \sqrt{V^2p^2 + c^2 - 2(p-2k)Vc} \quad (k = 1, 2, \ldots, p-1)$$
$$E_p^+ = pV + \left(1 - \tfrac{p}{2}\right)c.$$

Thus we have recovered the well known diagonalization of the Moszkowski Hamiltonian but by using one of the differential realizations of the Lie superalgebra $q(2)$. The latter can then be considered as a 'spectrum generating superalgebra' of the Moszkowski model.

7. Jaynes–Cummings model

The well known Jaynes–Cummings Hamiltonian [24] is one of the diagonalizable Hamiltonians of quantum optics. It describes a two-level atom interacting with a single-mode radiation. Under the so-called rotating wave approximation for which only real transitions (e.g. a photon is absorbed while the electron jumps from level 1 to level 2) are taken into account, the Jaynes–Cummings Hamiltonian is

$$H_{\mathrm{JC}} = \omega\left(a^+a^- + \tfrac{1}{2}\right) - \tfrac{1}{2}\omega_0\sigma_3 + g(a^-\sigma_- + a^+\sigma_+). \tag{60}$$

Here ω is the field mode frequency, ω_0 the atomic frequency while g is a real coupling constant and, as usual, a^- and a^+ denote the photon annihilation and creation operators, respectively.

In order to determine the spectrum of H_{JC}, one can use the irreducible representations of the Lie superalgebra $u(1, 1)$ as shown in [25]. We will prove in this section that the Lie superalgebra $q(2)$ can play a similar role and thus be considered as a 'spectrum generating superalgebra' for the Jaynes–Cummings Hamiltonian. For this purpose, we shall use the basis vectors (12)–(14) consisting of the states Λ_k $(k = 0, 1, \ldots, p)$ and χ_l $(l = 1, 2, \ldots, p-1)$. This time, however, we shall consider the following realization of these basis vectors:

$$\Lambda_k = \begin{pmatrix} px^{p-k} \\ (p-k)x^{p-k-1} \end{pmatrix} \qquad (k = 0, 1, \ldots, p)$$
$$\chi_l = \begin{pmatrix} 0 \\ x^{p-l-1} \end{pmatrix} \qquad (l = 1, 2, \ldots, p-1) \tag{61}$$

as opposed to (23). This new realization of the basis states leads to a third differential realization of the $q(2)$-generators given by

$$
\begin{aligned}
&b^- = -x^2\frac{\mathrm{d}}{\mathrm{d}x} + (p-1)x + x\sigma_3 + \sigma_- &&b^+ = \frac{\mathrm{d}}{\mathrm{d}x}\\
&e_{00}^{\bar 0} - e_{11}^{\bar 0} = 2x\frac{\mathrm{d}}{\mathrm{d}x} + 1 - p - \sigma_3 &&e_{00}^{\bar 0} + e_{11}^{\bar 0} = p\\
&f^- = \sqrt{p}(x\sigma_3 + \sigma_- - x^2\sigma_+) &&f^+ = \sqrt{p}\sigma_+\\
&e_{00}^{\bar 1} - e_{11}^{\bar 1} = \sqrt{p}(-\sigma_3 + 2x\sigma_+) &&e_{00}^{\bar 1} + e_{11}^{\bar 1} = \frac{2}{\sqrt{p}}\left(\frac{p}{2}\sigma_3 + \frac{\mathrm{d}}{\mathrm{d}x}\sigma_-\right).
\end{aligned}
\tag{62}
$$

It has to be noticed that the realization of the $sl(2)$ subalgebra generated by b^-, b^+ and $e_{00}^{\bar 0} - e_{11}^{\bar 0}$ as defined in (62) coincides with the one performed in [26], but with other arguments. Taking in the Hamiltonian (60) the realization

$$
a^+ = x \qquad a^- = \frac{\mathrm{d}}{\mathrm{d}x} \tag{63}
$$

we can express H_{JC} as

$$
\begin{aligned}
H_{\mathrm{JC}} = {}&\frac{\omega}{2}\left(e_{00}^{\bar 0} - e_{11}^{\bar 0}\right) + \frac{1}{2}p\omega + \frac{g}{2}\sqrt{p}\left(e_{00}^{\bar 1} + e_{11}^{\bar 1}\right)\\
&+ \frac{1}{2}\frac{g}{\sqrt{p}}\left(e_{00}^{\bar 1} - e_{11}^{\bar 1}\right) + \frac{1}{2}(\omega_0 - \omega + g(p-1))\sigma_3.
\end{aligned}
\tag{64}
$$

From this equation it is clear that the $q(2)$ superalgebra is a ‘spectrum generating superalgebra’ of the Jaynes–Cummings Hamiltonian *provided* the detuning Δ ($\equiv \omega - \omega_0$) satisfies

$$
\Delta = g(p-1). \tag{65}
$$

Suppose this is the case. Then the action of (64) on the basis elements Λ_k and χ_l follows from (15) and (16). In fact, Λ_0 and Λ_p are directly eigenvectors of H_{JC} (with the eigenvalues E_0^+ and E_p^+ respectively), whereas the other eigenvectors are simple linear combinations of Λ_k and χ_k ($k = 1, 2, \ldots, p-1$). Thus it is straightforward to recover the Jaynes–Cummings spectrum, i.e.

$$
\begin{aligned}
&E_0^+ = \omega p + \tfrac{1}{2}(p+1)g\\
&E_k^\pm = \omega(p-k) \pm g\sqrt{\tfrac{1}{4}p^2 + \tfrac{1}{2}p + \tfrac{1}{4} - k} \qquad (k = 1, 2, \ldots, p-1)\\
&E_p^+ = \tfrac{1}{2}(p-1)g
\end{aligned}
$$

where the positive integer p is arbitrary.

Acknowledgment

The authors thank Y Brihaye for some useful comments and for pointing out reference [22].

References

[1] Gol’fand Yu A and Lihtman E P 1971 *JETP Lett.* **13** 323
[2] Wess J and Zumino B 1974 *Nucl. Phys.* B **78** 1
[3] Salam A and Strathdee J 1974 *Nucl. Phys.* B **80** 499
[4] Kac V G 1977 *Adv. Math.* **26** 8
[5] Kac V G 1978 *Lecture Notes Math.* **676** 597
[6] Scheunert M 1979 *The Theory of Lie Superalgebras* (Berlin: Springer)
[7] Corwin L, Ne’eman Y and Sternberg S 1975 *Rev. Mod. Phys.* **47** 573
[8] Balantekin A B 1984 *J. Math. Phys.* **25** 2028
[9] Hurni J P 1987 *J. Phys. A: Math. Gen.* **20** 5755

[10] Heidenreich W 1982 *Phys. Lett.* B **110** 461
[11] Freedman D Z and Nicolai H 1984 *Nucl. Phys.* B **237** 342
[12] Flato M and Fronsdal C 1984 *Lett. Math. Phys.* **8** 159
[13] Dobrev V K and Petkova V B 1985 *Phys. Lett.* B **162** 127
[14] Van der Jeugt J 1987 *J. Math. Phys.* **28** 758
[15] Penkov I 1986 *Funct. Anal. Appl.* **20** 30
[16] Penkov I and Serganova V 1997 *Lett. Math. Phys.* **40** 147
[17] Penkov I and Serganova V 1997 *J. Math. Sci.* **84** 1382
[18] Palev T D and Van der Jeugt J 2000 *J. Phys. A: Math. Gen.* **33** 2527
[19] Klinkhamer F R and Manton N S 1984 *Phys. Rev.* D **30** 2212
[20] Brihaye Y, Kosinski P, Giller S and Kunz J 1992 *Phys. Lett.* B **293** 383
Brihaye Y and Kosinski P 1994 *J. Math. Phys.* **35** 3089
[21] Arscott F M 1964 *Periodic Differential Equations* (Oxford: Pergamon)
[22] Brihaye Y, Giller S, Kosinski P and Nuyts J 1997 *Comm. Math. Phys.* **187** 201
[23] Moszkowski S A 1958 *Phys. Rev.* **110** 403
Bonatsos D, Brito L and Menezes D 1993 *J. Phys. A: Math. Gen.* **26** 895
[24] Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
[25] Buzano C, Rasetti M G and Rastello M L 1989 *Phys. Rev. Lett.* **62** 137
[26] Zhdanov R Z 1997 *Phys. Lett.* B **405** 253